**Lecture-5**

**Complex Differentiation and**

**The Cauchy-Riemann Equation**

**Analytic Functions:**

If a single valued function is differentiable i.e. exists at every point of a domain *D* except possibly at a finite number of exceptional points then the function is said to be **analytic** in the domain *D*. These exceptional point at which does not exist are called **singular points** or **singularities of the function**.

**Necessary conditions for to be analytic:**

**RECTANGULAR FORM:**

If and satisfies the **Cauchy-Riemann equations(C-R)** i.e.,

 and 

i.e.,  and 

then ****is said to be **analytic.**

Hence**,** at points whereexists may be obtained from either of

 or 

**POLAR FORM:**

If and satisfies the **Cauchy-Riemann equations(C-R)** i.e.

 and 

i.e.,  and 

then ****is said to be **analytic.**

Hence, at points whereexists may be obtained from either of

****

**Important Formulae:**

|  |  |
| --- | --- |
|  |  |
|  |  |
|  |  |

**Example: 1**

Verify **C-R** equations for the function and hence find.

**Solution:** Given,

or,

or,

or,

or,

Here, and .

Now, partially differentiating and with respect to and, we get

From the above result, we can write

and

Since satisfies Cauchy-Riemann equations, so is analytic.

|  |
| --- |
| **Example: 2**  Verify **C-R** equations for the function and hence find . |
| **Solution:**  Given    or,  or,  or,  or,  Here , and  Partially differentiating and with respect to and, we get  From the above result, we can write  and .  Since satisfies Cauchy-Riemann equations, so is an analytic function.            . |

**Exercise set: 5.1**

1. Write Cauchy-Riemann (**C-R**) equations in rectangular and polar forms.
2. For the following functions:



1. separate real and imaginary parts,
2. verify **C-R**  equations,
3. find or .
4. Justify whether the following functions satisfy the (**C-R)** equations. If analytic, then find 



1. Justify whether the following functions satisfy the (**C-R)** equations. If analytic, then find 



**Mappings**

**Geometrical Representation:**

To draw curve of complex variable we take two axes i.e., one real axis and the other imaginary axis. A number of points are plotted on -plane, by taking different value of (different value of *x* and *y*). The curve C is drawn by joining the plotted points. The diagram obtained is called **Argand diagram.**

**Transformation:**

For every point in the z-plane, the relation defines a corresponding point in the -plane. We call this “transformation or mapping of -plane into -plane”. If a point maps into the point, is known as the image of.

If the point moves along a curve *C* in -plane, the point will move along a corresponding curve in the -plane. We, then, say that a curve C in the -plane is mapped into the corresponding curve in the - plane by the relation .

Translation, Rotation and reflection are the standard transformations. Terms such as **translation, rotation** and **reflection** are used to convey dominant geometric characteristics of certain mappings.

**Translation**

,

where,

Hence,

So, and

and

On substituting the values of and in the equation of the curve to be transformed we get the equation of the image in the -pane.

As an example the mapping where , can be thought of as a translation of each point of one unit to the right.

|  |  |
| --- | --- |
| **Example:**  Let the rectangular region in *z*-plane which is bounded by the lines    Determine the region of the *w*-plane into which is mapped under the transformation  . | |
| **Solution:**  Given  or, .  Hence and . | when ,    ,      *u* |

**Rotation:**

The mapping where and , can be thought of as a rotation of the radius vector for each non-zero point *z* through a right angle about the origin in the counter clock wise direction.

|  |  |
| --- | --- |
| **Example:**  Let the rectangular region inz-plane which is bounded by the lines    Determine the region of the *w*-plane into which is mapped under the transformation  . | |
| **Solution:**  Given  or, .  Hence and . | when ,    , |

**Reflection:**

The mapping transforms each point of into its reflection in the real axis.

|  |  |
| --- | --- |
| **Example:**  Let the rectangular region inz-plane which is bounded by the lines    Determine the region of the *w*-plane into which is mapped under the transformation  . | |
| **Solution:** | |
| Given  or, .  Hence and . | when ,    , |
|  |  |

**Example:**

Given triangle *T* in thez-plane with vertices at Determine the triangle of the *w*-plane into which is mapped under the transformation .

|  |  |
| --- | --- |
| **Solution:** | |
| Given  or, .  Hence and .  When | The vertices of the triangle are  Hence the sides are  and |

**Exercise Set-5.2**

1. Let the rectangular region in *z*-plane which is bounded by the lines

Determine the region of the *w*-plane into which is mapped under the following transformations:

(i) , (ii) ,

(iii) , (iv) ,

(v) .

2. Given triangle *T* in the *z* -plane with vertices at Determine the triangle of the *w*-plane into which is mapped under the following transformations:

(i) , (ii) ,

(iii) , (iv)